A NOTE ON ADDITIVITY OF POLYGAMMA FUNCTIONS

FENG QI AND BAI-NI GUO

Abstract. In the note, the functions $\left|\psi^{(i)}(e^x)\right|$ for $i \in \mathbb{N}$ are proved to be sub-additive on $(\ln\theta_i,\infty)$ and super-additive on $(-\infty,\ln\theta_i)$, where $\theta_i \in (0,1)$ is the unique root of equation $2\left|\psi^{(i)}(\theta)\right| = \left|\psi^{(i)}(\theta^2)\right|$.

1. Introduction

Recall [3, 5, 7] that a function f is said to be sub-additive on I if

$$f(x+y) \le f(x) + f(y) \tag{1}$$

holds for all $x, y \in I$ such that $x + y \in I$. If the inequality (1) is reversed, then f is called super-additive on I.

The sub-additive and super-additive functions play an important role in the theory of differential equations, in the study of semi-groups, in number theory, and also in the theory of convex bodies. A lot of literature for the sub-additive and super-additive functions can be found in [3, 5] and related references therein.

It is well-known that the classical Euler gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d} t. \tag{2}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. It is common knowledge that these functions are fundamental and important and that they have much extensive applications in mathematical sciences.

In [4], the function $\psi(a+x)$ is proved to be sub-multiplicative with respect to $x \in [0,\infty)$ if and only if $a \ge a_0$, where a_0 denotes the only positive real number which satisfies $\psi(a_0) = 1$.

In [5], the function $[\Gamma(x)]^\alpha$ was proved to be sub-additive on $(0,\infty)$ if and only if $\frac{\ln 2}{\ln \Delta} \le \alpha \le 0$, where $\Delta = \min_{x \ge 0} \frac{\Gamma(2x)}{\Gamma(x)}$.

In [2, Lemma 2.4], the function $\psi(e^x)$ was proved to be strictly concave on $\mathbb{R}$.

In [7, Theorem 3.1], the function $\psi(a+e^x)$ is proved to be sub-additive on $(-\infty,\infty)$ if and only if $a \ge c_0$, where c_0 is the only positive zero of $\psi(x)$.

In [6, Theorem 1], among other things, it was presented that the function $\psi^{(k)}(e^x)$ for $k \in \mathbb{N}$ is concave (or convex, respectively) on $\mathbb{R}$ if $k = 2n-2$ (or $k = 2n-1$, respectively) for $n \in \mathbb{N}$.

In this short note, we discuss sub-additive and super-additive properties of polygamma functions $\psi^{(i)}(x)$ for $i \in \mathbb{N}$.

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Our main result is the following Theorem 1.

Theorem 1. *The functions* $\left|\psi^{(i)}(e^x)\right|$ *for* $i\in\mathbb{N}$ *are super-additive on* $(-\infty,\ln\theta_i)$ *or sub-additive on* $(\ln\theta_i,\infty)$*, where* $\theta_i\in(0,1)$ *is the unique root of equation*

$$2\left|\psi^{(i)}(\theta)\right|=\left|\psi^{(i)}(\theta^2)\right|. \tag{3}$$

2. Proof of Theorem 1

Let

$$f(x,y)=\left|\psi^{(i)}(x)\right|+\left|\psi^{(i)}(y)\right|-\left|\psi^{(i)}(xy)\right| \tag{4}$$

for $x>0$ and $y>0$, where $i\in\mathbb{N}$. In order to show Theorem 1, it is sufficient to prove the positivity or negativity of the function $f(x,y)$. Direct differentiation yields

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x}&=y\left|\psi^{(i+1)}(xy)\right|-\left|\psi^{(i+1)}(x)\right|\\&=\frac{1}{x}\Big[xy\left|\psi^{(i+1)}(xy)\right|-x\left|\psi^{(i+1)}(x)\right|\Big].\end{aligned} \tag{5}$$

In [1, Lemma 1] and [8, 9], among other things, the functions $x^{\alpha}\left|\psi^{(i)}(x)\right|$ are proved to be strictly increasing on $(0,\infty)$ if and only if $\alpha\ge i+1$ and strictly decreasing if and only if $\alpha\le i$. From this monotonicity, it follows easily that $\frac{\partial f(x,y)}{\partial x}\gtreqless0$ if and only if $y\lesseqgtr1$, which means that the function $f(x,y)$ is strictly increasing for $y<1$ and strictly decreasing for $y>1$ in $x\in(0,\infty)$. Since

$$\lim_{x\to\infty}f(x,y)=\left|\psi^{(i)}(y)\right|>0,$$

then the function $f(x,y)$ is positive in $x\in(0,\infty)$ for $y>1$.

For $y<1$, by virtue of the increasing monotonicity of $f(x,y)$, it is deduced that

(1) if $x>1$, then $f(1,y)=\left|\psi^{(i)}(1)\right|<f(x,y)<\left|\psi^{(i)}(y)\right|$;
(2) if $x<1$, then $f(x,y)<f(1,y)=\left|\psi^{(i)}(1)\right|$;
(3) if $y<x<1$, then $f(y,y)<f(x,y)$;
(4) if $x<y<1$, then $f(x,x)<f(x,y)$.

This implies that

$$f(\theta,\theta)=2\left|\psi^{(i)}(\theta)\right|-\left|\psi^{(i)}(\theta^2)\right|<f(x,y) \tag{6}$$

for $y<1$, where $\theta<1$ with $\theta<x$ and $\theta<y$. Since $f(\theta,\theta)$ is strictly increasing on $(0,1)$ such that $f(1,1)=\left|\psi^{(i)}(1)\right|>0$ and $\lim_{\theta\to0^+}f(\theta,\theta)=-\infty$, then the function $f(\theta,\theta)$ has a unique zero $\theta_i\in(0,1)$ such that $f(\theta,\theta)>0$ for $1>\theta>\theta_i$.

In conclusion, the function $f(x,y)$ is positive for $x,y>\theta_i$ or negative for $0<x,y<\theta_i$. The proof of Theorem 1 is complete.

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(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: http://qifeng618.spaces.live.com

(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail address: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com
URL: http://guobaini.spaces.live.com